ALGEBRAIC SELF–MODELING SOLUTIONS OF TWO-DIMENSIONAL TRANSONIC GAS FLOW

(ALGEBRAICHESKIE AVTOMODEL'NYE RESHENIIA URAVNENII OKOLOZWUKOVOGO PLOSKOGO TECHENIIA GAZA)

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Approximate equations of the flow of gas in the transonic velocity range possess an important class of self-modeling solutions. Many of the transonic flow properties such as, for example, the character of flow at some distance from a body, the flow in Laval nozzles, etc., were analysed with such solutions as the main tool [1 to 3].

An analysis is made below of cases in which the self-modeling solutions are represented by algebraic functions. By resorting to parametric representation of the unknown magnitudes, it is possible to indicate in all cases a form of solution convenient for gas dynamical computations. Certain exact solutions of the Tricomi equation have been obtained in this manner, solutions which may be used in the analysis of new properties of transonic flows such as flow in a Laval nozzle with linked supersonic zones, flow in a nozzle with breaks in its wall, flow in the neighborhood of the intersection point of the sonic line with the sonic stream boundary, etc.

1. We shall consider a plane irrotational flow of a perfect compressible fluid in which the velocities are nearly sonic throughout. Such a flow is approximated in the hodograph plane by the system of equations of the form [4]

$$\frac{\partial \varphi}{\partial \theta} + \frac{\partial \psi}{\partial \eta} = 0, \qquad \frac{\partial \varphi}{\partial \eta} - \eta \frac{\partial \psi}{\partial \theta} = 0 \tag{1.1}$$

Here ψ is the stream function, φ is the velocity potential, η is a velocity function which becomes zero at the critical velocity, and θ is the angle of inclination of the velocity vector.

System (1.1) is equivalent to the single Tricomi equation of the stream function

$$\eta \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0 \tag{1.2}$$

Let us consider the self-modeling solutions of system (1.1) of the form [3]

$$\psi = \rho^k f(\xi), \quad \varphi = \rho^{k+1/3} g(\xi), \quad \rho = \sqrt{\theta^2 + {}^4/_9\, \eta^3}, \quad \xi = {}^4/_9\, \eta^3/\rho^2 \tag{1.3}$$

Putting ψ from the first equation of (1.3) into Equation (1.2), we obtain the following hypergeometric equation which defines the function f

$$\xi(1-\xi)f'' + [{}^2/_3 - {}^7/_6\xi]f' + {}^1/_2k\,({}^1/_2k + {}^1/_6)f = 0 \tag{1.4}$$

The parameters of this equation are:

$$a = -{}^1/_2k, \qquad b = {}^1/_2k + {}^1/_6, \qquad c = {}^2/_3 \tag{1.5}$$

It is convenient to present the solution of the hypergeometric equation in terms of the Riemann P-function which indicates the position of singular points of this equation, and the indices of singularities of this solution at such points

$$f = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & -{}^1/_2k & 0 & \xi \\ {}^1/_3 & {}^1/_2k + {}^1/_6 & {}^1/_2 & \end{matrix}\right\} \tag{1.6}$$

In the neighborhood of singular points function f can be found by means of series. Thus, in the neighborhood of the singular point $\xi = 1$, two linearly independent solutions of Equation (1.4) can be presented in the form

$$\begin{aligned} f_1 &= F(-{}^1/_2k,\ {}^1/_2k + {}^1/_6;\ {}^1/_2;\ 1-\xi) \\ f_2 &= \sqrt{1-\xi}\ \ F({}^1/_2 - {}^1/_2k,\ {}^2/_3 + {}^1/_2k;\ {}^3/_2;\ 1-\xi) \end{aligned} \tag{1.7}$$

The general solution of the hypergeometric equation contains two constant $f = C_1f_1 + C_2f_2$.

Since $1 - \xi = \theta^2 \ /\ \rho^{-2}$, it follows from (1.3) and (1.7) that the stream function ψ which corresponds to solution f_1 will be even in θ, while that pertaining to solution f_2, by virtue of the factor θ/ρ will be odd in θ. It is easy to see that solution f_2 defines flows which are symmetric about the x-axis in the physical plane. The flow in a Laval nozzle, and flows at some distance from a body belong to the latter category.

In order to find function g we substitute ψ and φ from (1.3) into Equations (1.1), and obtain two ordinary differential equations of first order with respect to f and g, from which we find

$$g = -\frac{6}{3k+1}\,\xi^{2/3}(1-\xi)^{1/2}f'(\xi) \tag{1.8}$$

Substituting for f its expression from (1.6), and using the formula for differentiating a hypergeometric function, we obtain for g the expression

$$g = -\frac{6}{3k+1}P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & -{}^1/_2k - {}^1/_6 & 0 & \xi \\ {}^2/_3 & {}^1/_2k & {}^1/_2 & \end{matrix}\right\} \tag{1.9}$$

2. We shall find for which values of k from (1.3), algebraic solutions of Equation (1.4) exist. We shall make use of the results obtained by Schwarz who had solved this problem for hypergeometric equations of the general type [6]. Firstly we shall find those values of k for which particular algebraic integrals of Equation (1.4) exist, when the second of the two linearly independent solutions may not be an algebraic function. This arises if, and only if one of the numbers $-{}^1/_2k$, ${}^1/_2k + {}^1/_6$, ${}^1/_2k + {}^2/_3$, and $-{}^1/_2k + {}^1/_2$ is an integer. This condition determines the following values

$$k = p, \qquad k = -{}^1/_3 + p \qquad (p = \pm 0,\ 1,\ 2,\ \ldots) \tag{2.1}$$

Corresponding solutions in the form of polynomials can be found in Guderley's book [3].

We shall now determine those values of k for which the general integral of the hypergeometric equation (1.4) is algebraic. Such cases occur if, and only if the exponent differences

$$\lambda = 1 - c, \quad \mu = a - b, \quad \nu = c - a - b \tag{2.2}$$

approximated to their integral parts, conform to the known Schwarz table [6]. For the equation (1.5) considered here we obtain the following five possibilities

$$\lambda = {}^1/_3, \quad \mu = {}^1/_2, \quad \nu = {}^1/_2 \quad \text{for} \quad k = {}^1/_3 + p \tag{2.3}$$

$$\lambda = {}^1/_3, \quad \mu = {}^1/_3, \quad \nu = {}^1/_2 \quad \text{for} \quad k = {}^1/_6 + p, \quad k = -{}^1/_2 + p \tag{2.4}$$

$$\lambda = {}^1/_3, \quad \mu = {}^1/_4, \quad \nu = {}^1/_2 \quad \text{for} \quad k = {}^1/_{12} + p, \quad k = -{}^5/_{12} + p \tag{2.5}$$

$$\lambda = {}^1/_3, \quad \mu = {}^1/_5, \quad \nu = {}^1/_2 \quad \text{for} \quad k = {}^1/_{30} + p, \quad k = -{}^{11}/_{30} + p \tag{2.6}$$

$$\lambda = {}^1/_3, \quad \mu = {}^2/_5, \quad \nu = {}^1/_2 \quad \text{for} \quad k = {}^7/_{30} + p, \quad k = -{}^{17}/_{30} + p \tag{2.7}$$

As was proved by Frankl [2], the value $k = 1/3$ from (2.3), leads to self-modeling solutions defining the flow in a Laval nozzle with a curved transition line. Falkovich had noted [4] that in this case the solution is in algebraic form and found the relevant general integral. Frankl [5] gave later the whole family (2.3) and the form of integrals, describing flows symmetric about the x-axis in the physical plane. With $k = -5/3$ from the same family we obtain a solution for the fundamental singularity of a flow at some distance from a body [1]. When $k = 4/3$, we have a particular integral which defines the flow around a corner. We may note, incidentally, that the relevant numerical solution derived by Vaglio-Laurin [7], was obtained in an analytical form by us in [8], where we also gave particular integrals which exist for the cases of $k = 1/6$ in (2.4), $k = 1/12$ in (2.5), and $k = 1/30$ in (2.6). Lifschitz and Ryzhov [9] have derived the same particular integrals for $k = 1/6$ and $k = 1/12$ in a different manner, and had indicated the families (2.4) and (2.5), however, their work dealt with particular algebraic integrals.

3. We shall use the Schwarz method [10] for the actual computation of the hypergeometric function appearing in (1.6). We reduce equation (1.4) to the normal form by substituting into it

$$h(\xi) = \xi^{1/3}(1 - \xi)^{1/4} f(\xi) \tag{3.1}$$

For the determination of h we obtain Equation

$$\frac{d^2h}{d\xi^2} + Ih = 0 \qquad \left(I = \frac{2}{9\xi^2} + \frac{3}{16(1-\xi)^2} + \frac{23 - (6k+1)^2}{144\xi(1-\xi)}\right) \tag{3.2}$$

Let f_1 and f_2 be two linearly independent particular integrals of equation (1.4), not necessarily congruent with solutions (1.7). To these correspond two integrals h_1 and h_2 of Equation (3.2). We introduce into our analysis the Schwarz function defined as the ratio of two particular integrals

$$s = f_1 : f_2 = h_1 : h_2 \tag{3.3}$$

We shall now derive the equation which is satisfied by function s. As each of the integrals f_1 and f_2 contain two arbitrary constants, function s, which is determined with an accuracy of the order of the multiplication constant, contains three constants. The equation for s will, therefore, be of the third order. Denoting differentiation with respect to ξ by primes, we can write

$$h_1'' + Ih_1 = 0, \qquad h_2'' + Ih_2 = 0 \tag{3.4}$$

Replacing h_1 in the first of above equations with sh_2, and taking into account the second equation of (3.4), we obtain

$$\frac{s''}{s'} = -2\frac{h_2'}{h_2} \tag{3.5}$$

Differentiating this equation with respect to ξ, and using the second equation of (3.4) together with Equation (3.5) for the elimination of h_2 and its derivatives, we obtain for s the expression

$$\frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 = 2I \tag{3.6}$$

If a particular solution of Equation (3.6) is found, then by integrating (3.5), we first obtain the solution $h_2 = (s')^{-1/2}$ of Equation (3.2), and then utilising (3.3), the second solution of this equation $h_1 = s\,(s')^{-1/2}$. The general solution of (3.4) is then of the form

$$h = (s')^{-1/2}(C_1 s + C_2) \tag{3.7}$$

Taking into account relationships (3.1) we can write down the generalised integral of the hypergeometric equation (1.4)

$$f = \left(\frac{d\xi}{ds}\right)^{1/2}\xi^{-1/3}(1-\xi)^{-1/4}(C_1 s + C_2) \tag{3.8}$$

4. Having found the solution in the hodograph plane, we must transpose it back onto the physical plane. Reverting to variables φ and ψ, we obtain instead of (1.1) the following system

$$\frac{\partial\eta}{\partial\psi} + \frac{\partial\theta}{\partial\varphi} = 0, \qquad \frac{\partial\theta}{\partial\psi} - \eta\frac{\partial\eta}{\partial\varphi} = 0 \tag{4.1}$$

We shall limit our considerations to the analysis of flows not much different from a plane parallel flow along the x-axis of the physical plane. We can then substitute in (4.1) x for ϕ, y for ψ, $-u$ for η, and v for θ, where u and v are the dimensionless components of the sonic stream perturbation velocity along the axes of the orthogonal coordinate system x, y. With this condition, system (4.1) becomes equivalent to the system of approximate equations derived by Karman for transonic flows [11].

In the physical plane the self-modeling solutions are expressed by [12 and 13]

$$u = y^{2(n-1)}U(\zeta), \qquad v = y^{3(n-1)}V(\zeta), \qquad \zeta = xy^{-n} \tag{4.2}$$

Here n is related to k of (1.3) by

$$n = \frac{3k+1}{3k} \tag{4.3}$$

Functions U and V satisfy the following system of differential equations [14]

$$u\zeta\frac{dn}{d\zeta} + \frac{dV}{d\zeta} - 2(n-1)U = 0, \qquad U\frac{dU}{d\zeta} - 3(n-1)V + n\zeta\frac{dV}{d\zeta} = 0 \tag{4.4}$$

We substitute for U and V variables as follows [15]

$$t = U\zeta^{-2}, \qquad \tau = V\zeta^{-3} \tag{4.5}$$

System (4.4) is now equivalent to

$$\frac{d\tau}{dt} = \frac{2(n-1)t^2 + 3t\tau - 3n\tau}{2t^2 - 2nt - 3(n-1)\tau}, \qquad \frac{d\zeta}{\zeta} = \frac{(n^2 - t)\,dt}{2t^2 - 2nt - 3(n-1)\tau} \tag{4.6}$$

These equations have the following singular points at finite distances from the origin

$$A\ (t=0,\ \tau=0),\quad C\ (t=n^2,\ \tau={}^2/_3 n^3),\quad D\ (t=1,\tau=-{}^2/_3) \tag{4.7}$$

Point A corresponds to the x-axis of the physical plane, while the singular point C represents the limit characteristic. Infinity in the $t\ \tau$-plane, which we shall denote by point B, corresponds to the y-axis.

Second equation of the system (4.6) shows that, when the integral curve $t = t\ (\tau)$ of the first equation of (4.6) intersects line $t = n^2$ at a point other than the singular point C, then $d\xi = 0$. This indicates the appearance of a limit line in the corresponding flow.

The t, τ-plane called the phase plane, is convenient for analysing flows in the presence of shock waves. Denoting parameters related to the two sides of a shock wave by indices 1 and 2, we write conditions at discontinuities as follows [16]

$$t_2 = 2n^2 - t_1,\qquad \tau_2 = \tau_1 + 2nt_1 - 2n^3,\qquad \zeta_1 = \zeta_2 \tag{4.8}$$

To determine the streamline we must integrate the approximate equation

$$\frac{dy^\circ}{dx} = v \qquad \text{for} \qquad y = \text{const} \tag{4.9}$$

We integrate the second equation of (4.1) over the area bounded by line $\zeta = \text{const}$, and lines $y = 0$, $x = -\infty$ and $y = y_0$. Using Green's formula, we obtain

$$\oint\left(\frac{u^2}{2}\,dy + v\,dx\right) = 0 \tag{4.10}$$

Along $y = y_0$ the first term of (4.10) is equal to zero, and the second one, to y_0. The integral can be easily computed along line $\zeta = \text{const}$, if for u and v we substitute their expressions from (4.3)

$$y^\circ = \frac{1}{4n-3}\,y_0^{4n-3}\,Y\ (\zeta) \qquad \left(Y = \frac{1}{2}U^2 + n\zeta V\right) \tag{4.11}$$

The analysis carried out in section 2 established all possible cases in which the nonlinear system (4.4) can be solved in terms of algebraic functions. Corresponding values of n can be easily found by resorting to the transformation formula (4.3), and to results obtained in (2.1) and (2.3) to (2.7).

5. We shall now find the general integral for one single value of k from each of the infinite families (2.4) to (2.7). Other hypergeometric functions of the same family can be found by consecutive differentiation of the function thus obtained. Family (2.3) is assumed known [5], and will not be considered.

We take $k = 1/6$ from family (2.4). For this value of k there exists a solution of the Schwarz equation [6] which is

$$\xi = \left[\frac{H_1(s)}{F_1(s)}\right]^3,\qquad \begin{array}{l} H_1(s) = s^4 + 2\sqrt{3}s^2 - 1 \\ F_1(s) = s^4 - 2\sqrt{3}s^2 - 1 \end{array} \tag{5.1}$$

Polynomials H_1 and F_1 are such that the following identity holds

$$H_1^3 - F_1^3 = 12\sqrt{3}\,[T_1(s)]^2,\qquad T_1(s) = s\,(s^4+1) \tag{5.2}$$

Using Expression (5.1) for ξ and the identity (5.2), we find

$$1 - \xi = -12\sqrt{3}\,\frac{T_1^2}{F_1^3} \tag{5.3}$$

Differentiation of (5.1) yields

$$\frac{d\xi}{ds} = -24\sqrt{3}T_1H_1^2F_1^{-4} \tag{5.4}$$

We shall now find function f by using Formula (3.8)

$$f = (C_1s + C_2)F_1^{-1/4} \tag{5.5}$$

Equations (5.1) and (5.5) together define parametrically in terms of s the hypergeometric function $f = f(\xi)$ which is the solution of Equation (1.4) for $k = 1/6$. We shall now derive the expression of $g = g(\xi)$ using Formula (1.8). Differentiating (5.5) with respect to s, we obtain

$$\frac{df}{ds} = -[M_1(s)]F_1^{-5/4}, \qquad M_1(s) = C_2s^3 - \sqrt{3}C_1s^2 + \sqrt{3}C_2s + C_1 \tag{5.6}$$

Using the relationship $df/d\xi = (df/ds)(ds/d\xi)$ and Equations (5.6), (5.4), (5.3) and (5.1), and substituting into (1.8) for ξ, $1-\xi$ and $df/d\xi$ their expressions in terms of s, we obtain

$$g = -3^{-3/4}M_1F_1^{-3/4} \tag{5.7}$$

In this case it is possible to derive the explicit expression for $f = f(\xi)$, and consequently for Function $g = g(\xi)$. In fact, Equation (5.1) can be solved with respect to Function s in the radical form [6]

$$s = -\sqrt{-i}\left(\frac{\delta\sqrt{1-\varepsilon\xi^{1/3}} - \delta^{-1}\sqrt{1-\varepsilon^{-1}\xi^{1/3}}}{\delta\sqrt{1-\varepsilon\xi^{1/3}} + \delta^{-1}\sqrt{1-\varepsilon^{-1}\xi^{1/3}}}\right)^{1/2}$$
$$\delta = \exp\frac{\pi i}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}}i, \qquad \varepsilon = \exp\frac{2\pi i}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \tag{5.8}$$

Substituting into (5.5) the expression of s from (5.8), we obtain the general integral of Equation (1.4). We shall find the explicit expressions for the particular solutions f_1 and f_2 given by

$$f_1 = F\left(-\frac{1}{12}, \frac{1}{4}; \frac{1}{2}; 1-\xi\right) = \frac{1}{\sqrt{3^{1/4}2}}\sqrt{\delta\sqrt{1-\varepsilon\xi^{1/3}} + \delta^{-1}\sqrt{1-\varepsilon^{-1}\xi^{1/3}}} \tag{5.9}$$

$$f_2 = \sqrt{1-\xi}\,F\left(\frac{5}{12}, \frac{3}{4}; \frac{3}{2}; 1-\xi\right) = \frac{\sqrt{3^{1/4}6}}{\sqrt{i}}\sqrt{\delta\sqrt{1-\varepsilon\xi^{1/3}} - \delta^{-1}\sqrt{1-\varepsilon^{-1}\xi^{1/3}}} \tag{5.10}$$

In this work, however, preference is given to the parametric presentation of all unknown functions, as this permits a complete solution of the problem, in a form convenient for gas-dynamic computations in all of the cases (2.4) to (2.7).

We shall derive the solution of system (4.4) for $k = 1/6$. The variable ζ defined by Expression (4.2) is found by using (1.3), (5.5) and (5.7)

$$\zeta = \frac{\varphi}{\psi^3} = G\frac{1 + \sqrt{3}Es - \sqrt{3}s^2 + Es^3}{(E+s)^3} \qquad \left(E = \frac{C_2}{C_1} = \text{const}, \quad G = \text{const}\right) \tag{5.11}$$

Taking into account substitution (4.5) we can rewrite Equations (4.2) as follows

$$t = \left(\frac{\psi}{\varphi}\right)^2 u, \qquad \tau = \left(\frac{\psi}{\varphi}\right)^3 v \tag{5.12}$$

If we substitute the previously found expressions of φ and ψ into the right-hand sides of these equalities, we obtain the general solution of the first of Equations (1.6) in a parametric form

$$t = -3\sqrt{3}\frac{(s^4 - 2\sqrt{3}s^2 - 1)(s + E)^2}{(1 + \sqrt{3}Es - \sqrt{3}s^2 + Es^3)^2}, \qquad \tau = 36\frac{s(s^4 + 1)(s + E)^3}{(1 + \sqrt{3}Es - \sqrt{3}s^2 + Es^3)^3} \tag{5.13}$$

Functions U and V are determined from Formula (4.5) by using the expression for ζ given in (5.11)

$$U = -3\sqrt{3}G^2\frac{s^4 - 2\sqrt{3}s^2 - 1}{(E + s)^4}, \qquad V = 36\, G^3\frac{s(s^4 + 1)}{(E + s)^6} \tag{5.14}$$

Equations (5.14) together with (5.11) determine the parametric solutions $U = U(\zeta)$, and $V = V(\zeta)$ of system (4.4).

In their analysis of the problem of an asymptotic type of plane parallel flow in the neighborhood of the center of a Laval nozzle, Lifschitz and Ryzhov [14] had considered a nozzle corresponding to the value $n = 3$. It is possible to write down this solution in its final form. In a hodograph plane this is given by the function f_2 which was defined in (5.10). We shall derive the corresponding solution in the physical plane. A flow which is symmetric about the x-axis is defined in the hodograph plane by the following expansion of Equation (4.6) in the neighborhood of the singular point A (4.7)

$$\tau = \frac{2(n-1)}{n}t^2 + \frac{2}{3}\frac{(1-n)(12n^2 - 25n + 12)}{n^3}t^3 + \ldots \tag{5.15}$$

In the case of $n = 3$, this expansion is obtained from (5.13), on the assumption that $E = 0$

$$t = -\frac{3\sqrt{3}(s^4 - 2\sqrt{3}s^2 - 1)s^2}{(1 - \sqrt{3}s^2)^2}, \qquad \tau = 36\frac{(s^4 + 1)s^4}{(1 - \sqrt{3}s^2)^3} \tag{5.16}$$

The corresponding curve is shown on fig. 1. Point A in the $t\tau$-plane, and the x-axis in the physical plane, correspond to the value of parameter $s = 0$. By assigning to parameter s increasing real values we move in the $t\tau$-plane along curve 1 in the direction of $t > 0$. When $s\ (2-3^{1/2})^{1/2}$, the singular point C is reached. A further increase of s to $s = 3^{-1/4}$ yields curve CB_1 which stretches to infinity, and then reappears from the direction of $\tau < 0$ (point B_2). At $s = 3^{1/4}$ the curve intersects line $t = 9$ at the point L which indicates a limit line. When $s \to \infty$, the integral curve $t = t(\tau)$ becomes infinite at the point B_3, with its asymptotic behavior defined by $\tau = 4t$. It can be easily shown that in the case of an arbitrary index of self-modeling, if the integral curve behaves asymptotically in the $t\tau$-plane as

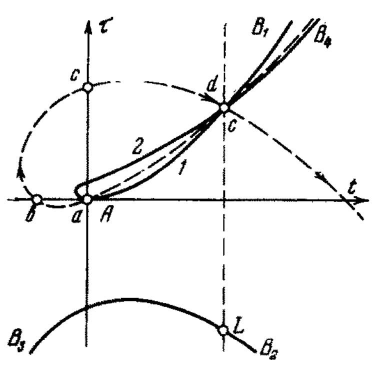

FIG. 1

$$\tau \sim 2(n-1)t \quad \text{as} \quad t \to \infty \tag{5.17}$$

then the corresponding analytical flow in the physical plane will be symmetric about the y-axis. It appears that when $n = 3$, a flow symmetric about the x-axis is also symmetric about the y-axis. Therefore, an analytic continuation beyond the y-axis results in the above curve in the $t\tau$-plane being retraced in the reverse order, namely $B_3LB_2B_1C1A$. From a physical point of view the corresponding flow is of little interest, because of the presence of a triple coverage of the physical plane which cannot be eliminated by the introduction of discontinuities.

Equations (5.16) show that it is possible to plot a real curve $t = t\,(\tau)$ by assigning to parameter s not only real, but also purely imaginary values. We introduce the notation $s = is_1$ and substitute this into Formulas (5.16)

$$t = \frac{3\sqrt{3}\,(s_1^4 + 2\sqrt{3}s_1^2 - 1)}{(1+\sqrt{3}s_1^2)^2}, \qquad \tau = \frac{36\,(1+s_1^4)\,s_1^4}{(1+\sqrt{3}s_1^2)^3}$$

The behavior of this curve is shown on fig. 1. With $s_1 = 0$ we have point A. With increasing s the curve runs from this point in the direction of $t < 0$, and then intersects the line $t = 0$, which indicates that sonic velocity is reached when $s_1 = \sqrt{2-\sqrt{3}}$. From here, we follow curve 2, and reach point C, when $s_1 = \sqrt{2+\sqrt{3}}$. With further increase of s_1 we plot curve CB_4. When $s_1 \to \infty$, then $\tau \sim 4t$. This indicates a flow symmetric about the y-axis. With an analytic continuation beyond the y-axis, we move along the plotted curve in the opposite direction, namely B_4C2A.

Let us construct the corresponding flow. We specify $E = 0$, $s = is_1$ and $G = iG_1$ in Formulas (5.11) and (5.14), and obtain

$$\zeta = -G_1\,\frac{1+\sqrt{3}s_1^2}{s_1^3}, \quad U = 3\sqrt{3}G_1^2\,\frac{s_1^4+2\sqrt{3}s_1^2-1}{s_1^4}, \quad V = -36G_1^3\,\frac{s_1^4+1}{s_1^5} \qquad (5.18)$$

These formulas define functions $U = U\,(\zeta)$ and $V = V\,(\xi)$ which characterize the magnitude of velocity components u and v along the straight line $y = \mathrm{const}$ in the left-hand half-plane $\zeta \leqslant 0$. In the right-hand half-plane we have to use expressions (5.18) and assume $G_1 = -G_2$. These functions are shown on fig. 2, where the branch indexed 2 should

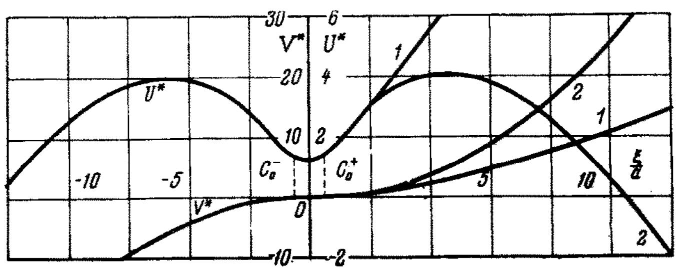

FIG. 2

be taken in the right-hand half-plane. The plotting of streamlines in accordance with formula (4.11) shows that we have found a flow in a Laval nozzle, possessing two planes of symmetry, x and y. At the entry the flow is subsonic, then sonic velocity is reached, the stream is accelerated, then it stagnates in the critical section, then accelerates again, and finally smoothly changes to subsonic. This is the limiting flow in a Laval nozzle with local supersonic zones linked together at the x-axis of the nozzle. Fig. 3 shows the wall contour and lines $u = \mathrm{const}$, with C_0^- being the limiting characteristic upstream of the nozzle center, and C_0^+ downstream of it. It is important to note that this flow is analytical throughout, except at the coordinate origin, where there is a singularity which indicates the convergence point of supersonic zones. We find that the distribution of the longitudinal component of the velocity u along the axis of the nozzle is $u = -\mathrm{const}\,x^{4/3}$ from which we can see that for $x = 0$ the second derivative d^2u/dx^2 becomes infinite. We may note that similar flows considered in the work of Tomotiki and Tamada [17], and in that of Ryzhov [18] are not analytical, neither along the C_0^- characteristic upstream of the nozzle center, nor along the characteristic C_0^+ downstream of it, and can only be obtained with nozzles of special form, while the flow shown on fig. 3 will obtain apparently in any nozzle in the neighborhood of its center.

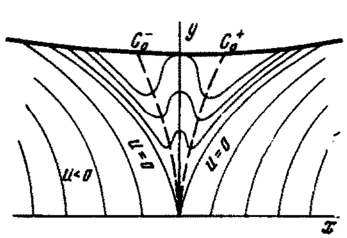

FIG. 3

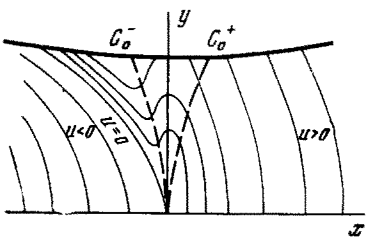

FIG. 4

Besides the flow shown on fig. 3 a wide range of Laval nozzle flows with weak discontinuities along the C_0^-, and C_0^+ characteristics and shock waves, can be analysed for $n = 3$, as was done by Ryzhov for $n = 2$ in [18]. We shall consider one of such flows. Let $A2CB_4B_4C1A$ be the representative curve in the $t\tau$-plane. This means that at the entry, up to the C_0^+ characteristic, the flow coincides with that analysed above, while along the C_0^+ characteristic there exists a weak discontinuity, beyond which the flow is supersonic right up to the axis, and is symmetric about the x-axis. Along the C_0^+ characteristic, a discontinuity of third derivatives of components of perturbation velocity u and v exists. Function U and V of this new flow are shown in fig. 2, where the branch indexed 1 is to be taken in the zone of $\zeta > \zeta_{C_0^+}$. Form of nozzle walls and lines $u = \text{const}$ are shown on fig. 4. This flow was analysed by Lifschitz and Ryzhov [14].

With the aid of the derived solutions of Tricomi's problem we shall analyse the change of various modes of gas flow through a Laval nozzle symmetric about the O_y-axis, which coincides with the critical section.

We shall consider the flow in a nozzle symmetric about the y-axis which coincides with the critical section. At subsonic velocities the field of flow in this nozzle is also symmetric about the y-axis. During the acceleration of gas, local supersonic zones which increase with increasing rate of output appear on the two sides of the critical section. Away from these zones, the flow remains symmetric about the y-axis, while in the zones themselves there may appear, generally speaking, shock waves which will upset the symmetry. In the limiting case, when the supersonic zones link together, as is shown on fig. 3, symmetry of the flow may still be preserved, but in the next moment, as shown on fig. 4, we have a transition to a Laval nozzle flow pattern with an abrupt disturbance of symmetry. After that, linked supersonic zones should gradually disappear, and the flow should revert to one, which is analytical at the nozzle center, and which corresponds to the value $n = 2$. It can be assumed that the flow shown on fig. 4 is unstable. A strict proof of the above would require the solution of equations of a two-dimensional, non-stationary supersonic flow.

As another important flow defined by a self-modeling solution for $n = 3$, we shall note the one which occurs in the neighborhood of the intersection point of a sonic stream boundary with the sonic line. Fig. 5 shows a stream with the critical velocity at its boundary, and a body placed in it. The flow upstream of the body and up to it is subsonic, then it is accelerated and becomes supersonic downstream of the body, with Oa being the sonic line, Ob the line of horizontal inclination of velocity, Oc the second sonic line, Od the limit

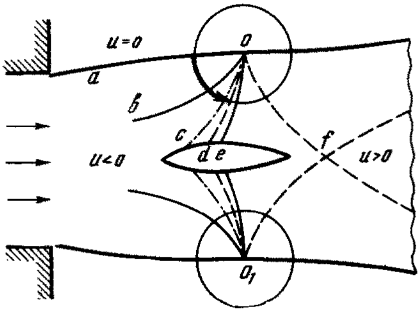

FIG. 5

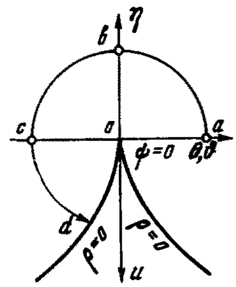

FIG. 6

characteristic C_0^+, Oe the line of horizontal inclination of velocity, and Of the limit characteristic C_0^-.

We shall consider the neighborhood of point O in the hodograph plane (fig. 6), where lines corresponding to those of fig. 5 are denoted by the same letters. A characteristic feature of the flow considered here is, that the streamline Oa is at the same time the sonic line on which the condition $\eta = 0$ is fulfilled. In the hodograph plane this condition is formulated thus: $\psi = 0$ on line Oa. In the neighborhood of $\eta = 0$ the hypergeometric equation (1.4) has a particular solution defined by the series

$$f_3 = \frac{\eta}{(^3/_2\,\rho)^{^2/_3}}\, F\left(-\frac{k}{2}+\frac{1}{3},\ \frac{k}{2}+\frac{1}{2};\ \frac{4}{3};\ \frac{4}{9}\,\frac{\eta^3}{\rho^2}\right) \tag{5.19}$$

Solution f_3 defines the stream function ψ which fulfils the boundary condition $\psi = 0$ for $\eta = 0$. We now continue f_3 (5.19) analytically into the neighborhood of the limit characteristic Od, as shown in fig. 6, and stipulate the regularity of the stream function ψ there. The analytical continuation of f_3 into the neighborhood of line Ob is given by Formula [19]

$$f_3 = Q_1 \frac{\eta}{(^3/_2\,\rho)^{^2/_3}}\, F\left(-\frac{k}{2}+\frac{1}{3},\ \frac{k}{2}+\frac{1}{2};\ \frac{1}{2};\ \frac{\theta^2}{\rho^2}\right) + $$
$$+\, Q_2 \frac{\eta}{(^3/_2\,\rho)^{^2/_3}}\, \frac{\theta}{\rho}\, F\left(\frac{k}{2}+1,\, -\frac{k}{2}+\frac{5}{6};\ \frac{3}{2};\ \frac{\theta^2}{\rho^2}\right) \tag{5.20}$$

$$Q_1 = \frac{\Gamma(^4/_3)\,\Gamma(^1/_2)}{\Gamma(^1/_2\,k+1)\,\Gamma(-^1/_2\,k+^5/_6)}, \qquad Q_2 = \frac{\Gamma(^4/_3)\,\Gamma(-^1/_2)}{\Gamma(-^1/_2\,k+^1/_3)\,\Gamma(^1/_2\,k+^1/_2)}$$

We shall then make use of relationships required for the analytical continuation into the neighborhood of Oc

$$F\left(-\frac{k}{2}+\frac{1}{3},\ \frac{k}{2}+\frac{1}{2};\ \frac{1}{2};\ \frac{\theta^2}{\rho^2}\right) = D_1 F\left(-\frac{k}{2}+\frac{1}{3},\ \frac{k}{2}+\frac{1}{2};\ \frac{4}{3};\ \frac{4}{9}\,\frac{\eta^3}{\rho^2}\right) + \tag{5.21}$$
$$+\, D_2 \frac{(^3/_2\,\rho)^{^2/_3}}{\eta}\, F\left(\frac{k}{2}+\frac{1}{6},\ -\frac{k}{2};\ \frac{2}{3};\ \frac{4}{9}\,\frac{\eta^3}{\rho^2}\right)$$

$$F\left(\frac{k}{2}+1,\ -\frac{k}{2}+\frac{5}{6};\ \frac{3}{2};\ \frac{\theta^2}{\rho^2}\right) = -D_3\,\frac{\rho}{\theta}\, F\left(\frac{1}{3}-\frac{k}{2},\ \frac{k}{2}+\frac{1}{2};\ \frac{4}{3};\ \frac{4}{9}\,\frac{\eta^3}{\rho^2}\right) -$$
$$-\, D_4 \frac{(^3/_2\,\rho)^{^2/_3}}{\eta}\,\frac{\rho}{\theta}\, F\left(\frac{k}{2}+\frac{1}{6},\ -\frac{k}{2};\ \frac{2}{3};\ \frac{4}{9}\,\frac{\eta^3}{\rho^2}\right) \tag{5.22}$$

$$D_1 = \frac{\Gamma(1/2)\,\Gamma(-1/3)}{\Gamma(1/2\,k + 1/6)\,\Gamma(-1/2\,k)}, \qquad D_2 = \frac{\Gamma(1/2)\,\Gamma(1/3)}{\Gamma(-1/2\,k + 1/3)\,\Gamma(1/2\,k + 1/2)}$$

$$D_3 = \frac{\Gamma(3/2)\,\Gamma(-1/3)}{\Gamma(-1/2\,k + 1/2)\,\Gamma(1/2\,k + 2/3)}, \qquad D_4 \frac{\Gamma(3/2)\,\Gamma(1/3)}{\Gamma(1/2\,k + 1)\,\Gamma(-1/2\,k + 5/6)}$$

The substitution of functions (5.21) and (5.22) into the right-hand side of Equation (5.20) yields for the neighborhood of line Oc

$$f_3 = \frac{\eta E_1}{(3/2\,\rho)^{2/3}} F\left(-\frac{k}{2} + \frac{1}{3}; \frac{k}{2} + \frac{1}{2}; \frac{4}{3}; \frac{4}{9}\frac{\eta^3}{\rho^2}\right) + E_2 F\left(\frac{k}{2} + \frac{1}{6}, -\frac{k}{2}; \frac{2}{3}; \frac{4}{9}\frac{\eta^3}{\rho^2}\right) \tag{5.23}$$

$$E_1 = Q_1 D_1 - Q_2 D_3, \qquad E_2 = Q_1 D_2 - Q_2 D_4$$

It remains now to carry out the continuation into the neighborhood of the limit characteristic Od. This is done by resorting to Formulas [19]

$$F\left(-\frac{k}{2} + \frac{1}{3}; \frac{k}{2} + \frac{1}{2}; \frac{4}{3}; \frac{4}{9}\frac{\eta^3}{\rho^2}\right) =$$
$$= R_1\left(-\frac{4}{9}\frac{\eta^3}{\rho^2}\right)^{\frac{k}{2} - \frac{1}{3}} F\left(\frac{1}{3} - \frac{k}{2}; -\frac{k}{2}; -k + \frac{5}{6}; \frac{9}{4}\frac{\rho^2}{\eta^3}\right) +$$
$$+ R_2\left(-\frac{4\eta^3}{9\rho^2}\right)^{-\frac{k}{2} - \frac{1}{2}} F\left(\frac{k}{2} + \frac{1}{2}, \frac{k}{2} + \frac{1}{6}; k + \frac{7}{6}; \frac{9\rho^2}{4\eta^3}\right) \tag{5.24}$$

$$F\left(\frac{k}{2} + \frac{1}{6}, -\frac{k}{2}\ \frac{2}{3}; \frac{4}{9}\ \frac{\eta^3}{\rho^2}\right) =$$
$$= R_3\left(-\frac{4}{9}\frac{\eta^3}{\rho^2}\right)^{-\frac{k}{2} - \frac{1}{6}} F\left(\frac{k}{2} + \frac{1}{2}, \frac{k}{2} + \frac{1}{6}; k + \frac{7}{6}; \frac{9\rho^2}{4\eta^3}\right) +$$
$$+ R_4\left(-\frac{4\eta^3}{9\rho^2}\right)^{\frac{k}{2}} F\left(-\frac{k}{2}, -\frac{k}{2} + \frac{1}{3}; -k + \frac{5}{6}; \frac{9\rho^2}{4\eta^3}\right) \tag{5.25}$$

$$R_1 = \frac{\Gamma(4/3)\,\Gamma(k + 1/6)}{\Gamma(1/2\,k + 1/2)\,\Gamma(1/2\,k + 1)}, \qquad R_2 = \frac{\Gamma(4/3)\,\Gamma(-k - 1/6)}{\Gamma(1/3 - 1/2\,k)\,\Gamma(-1/2\,k + 5/6)}$$

$$R_3 = \frac{\Gamma(2/3)\,\Gamma(-k - 1/6)}{\Gamma(-1/2\,k)\,\Gamma(-1/2\,k + 1/2)}, \qquad R_4 = \frac{\Gamma(2/3)\,\Gamma(k + 1/6)}{\Gamma(1/2\,k + 1/6)\,\Gamma(1/2\,k + 2/3)}$$

We substitute functions derived in (5.24) and (5.25) into Equation (5.23), and write down the expression of the stream function ψ (1.3) corresponding to solution f_3

$$\psi = N_1\left(-\frac{4}{9}\eta^3\right)^{\frac{k}{2}} F\left(-\frac{k}{2}, -\frac{k}{2} + \frac{1}{3}; -k + \frac{5}{6}; \frac{9\rho^2}{4\eta^3}\right) +$$
$$+ N_2\left(-\frac{4}{9}\eta^3\right)^{-\frac{k}{2} - \frac{1}{6}} \rho^{2k + 1/3} F\left(\frac{k}{2} + \frac{1}{6}, \frac{k}{2} + \frac{1}{2}; k + \frac{7}{6}; \frac{9\rho^2}{4\eta^3}\right) \tag{5.26}$$

Here

$$N_1 = -Q_1 D_1 R_1 + Q_2 D_3 R_1 + Q_1 D_2 R_4 - Q_1 D_4 R_4$$
$$N_2 = -Q_1 D_1 R_2 + Q_2 D_3 R_2 + Q_1 D_2 R_3 - Q_2 D_4 R_3$$

By substituting for Q, D and R their expressions from (5.20) to (5.25), and using the known Γ-function formula, $\Gamma(z)\Gamma(z+1) = \pi/\sin\pi x$, the coefficients N_1 and N_2 can be expressed by

$$N_1 = \frac{4}{3}\frac{\Gamma(^1/_3)\Gamma(k+^1/_6)}{k\Gamma(^1/_2 k)\Gamma(^1/_2 k+^1/_2)}\cos\pi k$$
$$N_2 = \frac{2}{3}\frac{\Gamma(^1/_3)\Gamma(-k-^1/_6)}{\Gamma(-^1/_2 k+^5/_6)\Gamma(-^1/_2 k+^1/_3)}\cos\pi(k+^1/_3) \tag{5.27}$$

If in Expression (5.26) the second term coefficient N_2 is equal to zero, then the stream function ψ is regular in the neighborhood of the limit characteristic defined by the equality $\rho = 0$. We obtain the following values of k

$$k = \frac{p}{2} - \frac{1}{3}\ (p = \pm 0, 1, 2, \ldots) \tag{5.28}$$

In particular $k = 1/6$ which corresponds according to Formula (4.3) to $n = 3$.

It is easy to prove that the solution f_3 given in (5.19) is determined in the neighborhood of point A in the $t\tau$-plane by the expansion

$$\tau = \frac{n}{3n-3}t + \frac{(3n-2)(2n-3)}{4n(3n-3)}t^2 + \frac{(3n-2)^2(2n^2-5n+2)}{56n^3(n-1)}t^3 + \ldots \tag{5.29}$$

For $n = 3$ this expansion yields that particular solution which is defined by Formulas (5.13) when $E = \sqrt{2+\sqrt{3}}$. Eliminating s, we obtain the following relationship

$$(\tau - t - 9)^2 = (1 + {}^1/_3 t)(t-9)^2 \tag{5.30}$$

This solution belongs to the class of solutions indicated in [8]. A similar solution was also obtained by Barantsev [20]. The relevant curve is shown on fig. 1 by a dotted line. Velocities are found with the aid of

$$U = G\zeta + {}^1/_{12}G^2, \qquad V = {}^1/_2 G\zeta^2 + {}^1/_3 G^2\zeta + {}^1/_{72}G^3 \tag{5.31}$$

6. We shall consider now the value $k = 1/12$ from the family (2.5). Following solutions of equation (3.6) exists for this value of k [6].

$$\xi = \frac{[H_2(s)]^3}{2^2 3^3 [F_2(s)]^4}, \qquad H_2(s) = 1 + 14s^4 + s^8, \qquad F_2(s) = s - s^5 \tag{6.1}$$

In the following computations we shall use the identity

$$H_2^3 - 2^2 3^3 F_2^4 = [T_2(s)]^2 \qquad (T_2(s) = 1 - 33s^4 - 33s^8 + s^{12}) \tag{6.2}$$

Using (6.1) and (6.2) we find Expression

$$1 - \xi = -2^{-2}3^{-3}\,T_2 F_2^{-4} \tag{6.3}$$

Differentiating (6.1) we obtain the derivative

$$\frac{d\xi}{ds} = -3^{-3}H_2^2 T_2 F_2^{-5} \tag{6.4}$$

Substituting for ξ, $1-\xi$, and $d\xi/ds$ their expressions from (6.1), (6.3) and (6.4) respectively, we obtain for f

$$f = (C_1 s + C_2)F_2^{-1/6} \tag{6.5}$$

The corresponding solution for g is derived from Formula (1.8)

$$g = -5^{-1}\cdot 2^{-1/3}\,3^{-1/2}\,F_2^{-5/6}\,M_2(s) \qquad M_2(s) = C_2 - 5C_1 s - 5C_2 s^4 + C_1 s^5 \tag{6.6}$$

Equations (6.5) and (6.6) together with (6.1) yield the parametric functions $f = f(\xi)$ and $g = g(\xi)$ which define the solution in the hodograph plane.

We may note that the solution considered in this section could also have been obtained from the solution derived in the preceding section by a quadratic transformation of the hypergeometric function. In fact, the particular solutions of (1.7) for $k = 1/12$ are of the form

$$f_1 = F(-{}^1/_{24}, {}^5/_{24}; {}^1/_2; 1-\xi), \qquad f_2 = \sqrt{1-\xi}\, F({}^{11}/_{24}, {}^{17}/_{24}; {}^3/_2, 1-\xi) \tag{6.7}$$

Using Goursat's table of quadratic transformations [21], we obtain

$$\frac{2\Gamma({}^1/_2)\Gamma({}^2/_3)}{\Gamma({}^{11}/_{24})\Gamma({}^{17}/_{24})} F\left(-\frac{1}{24}, \frac{5}{24}; \frac{1}{2}; 1-\xi\right) = F\left[-\frac{1}{12}, \frac{5}{12}; \frac{2}{3}; \frac{1}{2}(1+\sqrt{1-\xi})\right] + F\left[-\frac{1}{12}, \frac{5}{12}; \frac{2}{3}; \frac{1}{2}(1-\sqrt{1-\xi})\right]$$

$$\frac{2\Gamma(-{}^1/_2)\Gamma({}^2/_3)}{\Gamma(-{}^1/_{24})\Gamma({}^5/_{24})} \sqrt{1-\xi}\, F\left(\frac{11}{24}, \frac{17}{24}; \frac{3}{2}; 1-\xi\right) = F\left[-\frac{1}{12}, \frac{5}{12}; \frac{2}{3}; \frac{1}{2}(1-\sqrt{1-\xi})\right] - F\left[-\frac{1}{12}; \frac{5}{12}; \frac{2}{3}; \frac{1}{2}(1+\sqrt{1-\xi})\right] \tag{6.8}$$

The right-hand sides of Equations (6.8) contain function $F(-{}^1/_{12}, {}^5/_{12}, {}^2/_3, w)$, but using

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & -{}^1/_{12} & 0 & w \\ {}^1/_3 & {}^5/_{12} & {}^1/_3 & \end{Bmatrix} = w^{{}^1/_{12}} P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & -{}^1/_{12} & 0 & \dfrac{w-1}{w} \\ {}^1/_3 & {}^1/_4 & {}^1/_2 & \end{Bmatrix}$$

we return to function f as defined by (1.6) for $k = 1/6$, but with a changed argument. This, together with the conclusion reached in the preceding section as to the possibility of deriving the explicit form of function $f = f(\xi)$ for $k = 1/6$ shows, that such a form can also be found for the case $k = 1/12$.

We shall now obtain the solution of system (4.) for $k = 1/12$. We find variable ζ by using the results obtained in (6.4) and (6.6)

$$\zeta = G\, \frac{E - 5s - 5Es^4 + s^5}{(E+s)^5} \tag{6.9}$$

We write the general solution of Equation (4.7) in a parametric form

$$t = 5^2 \frac{(1 + 14s^4 + s^8)(E+s)^2}{(E - 5s - 5Es^4 + s^5)^2}, \qquad \tau = \frac{2}{3}\, 5^3\, \frac{(1 - 33s^4 - 33s^8 + s^{12})(E+s)^3}{(E - 5s - 5Es^4 + s^5)^3} \tag{6.10}$$

and find the solution of system (4.4) from Equation (6.9) together with Equations

$$U = (5G)^2 \frac{1 + 14s^4 + s^8}{(E+s)^8}, \qquad V = \frac{2}{3}(5G)^3 \frac{1 - 33s^4 - 33s^8 + s^{12}}{(E+s)^{12}} \tag{6.11}$$

We shall now find the value of E in Equation (6.10) which for $n = 5$ would yield a solution coinciding with expansion (5.15), and which defines a flow symmetric about the x-axis in the physical plane. It is easily seen that we must select for E a value equal in magnitude to one of the roots of the equation $T_2(s) = 0$, but of the opposite sign. Function $T_2(s)$ defined by (6.2) appears in the numerator of the expression of τ in (6.10). Let us find the real roots of equation $T_2(s) = 0$

$$s_{1,2} = 1 \pm \sqrt{2}, \qquad s_{3,4} = -1 \pm \sqrt{2} \tag{16.2}$$

Let $E = -(1 + \sqrt{2})$. We substitute this value of E into Equations (6.9) to (6.11), and plot the corresponding curve on the $t\tau$-plane. The behavior of this curve is shown schematically on fig. 7.

With $s = 1 + \sqrt{2}$ we have point A, and then, with the decreasing parameter, move in the direction of $t > 0$ along the integral curve defined by the first of equations (4.6). We reach point C at $s = 1$. When s passes through the root of the polynomial $M_2(s)$, $s = 0.840$, the curve becomes infinite along B_1, and then reappears on the side of $\tau < 0$ along B_2. With $s = 0.655$ the curve intersects line $t = 25$ (point L_1), then line $\tau = 0$, and at $s = 0$, we reach point C for the second time. With s approaching $s = s_2 = 1 - \sqrt{2}$, the curve stretches to infinity along B_3 with its asymptotic behavior defined by $t \sim 8t$. A further decrease of the parameter results in the same curve being traversed in the opposite direction, namely $B_3CL_1B_2B_1CA$. This curve can obviously be also extended into the area of $t < 0$. However, such an extension would necessitate the consideration not only of the real values of parameter s, but also those of the complex values for which the functions ζ, U, V, t and τ have real values. In the preceding section, both real and purely imaginary values were assigned to parameter s, when flows symmetric about the x-axis with $n = 3$ were considered. The question arises, which path is to be followed on the complex plane of parameter s in order to obtain all of the real values of the functions under consideration. In order to answer this question we shall turn to the following property of the Schwarz function [6]: the Schwarz function $s = s(\xi)$, defined by Equation (3.3), yields a conformal representation of the lower half-plane of variable ξ on the inside of a triangle delineated by circular arcs, with its inner angles equal to $\lambda\pi$, $\mu\pi$ and $\nu\pi$, where λ, μ and ν are defined by (2.2). Because only real values of ξ are considered here, and since in the case of self-modeling solutions all the gas dynamic parameters are expressed by ξ, it is clearly unnecessary to go beyond the area of this triangle in the complex s-plane. In order to extend curve $t = t(\tau)$ beyond the point A it will be necessary to move, in this case, on the parameter plane from the value of $s = 1 + \sqrt{2}$ along the circumference of a circle with its center at the point $s = 1$, and of radius $\sqrt{2}$.

The use of complex parameters is not convenient for computations. It is preferable to resort to a linear transformation which would transform the above circle into a real axis, by using the property of the Schwarz function, that its linear transformation also yields a solution of the Schwarz equation [10]. Denoting the new function again by s, we obtain along with solution (6.1) of Equation (3.6) for $k = 1/12$, the following solution

$$\xi = \frac{64\sqrt{2}\,[H_3(s)]^3}{[F_3(s)]^4} \qquad \begin{pmatrix} H_3(s) = s^7 - {}^1/_4\, 7\sqrt{2}\, s^4 - s \\ F_3(s) = s^6 + 5\sqrt{2}\, s^3 - 1 \end{pmatrix} \tag{6.13}$$

Instead of the identity (6.2) we shall have

$$\begin{gathered} F_3{}^4 - 64\sqrt{2}\, H_3{}^3 = [T_3(s)]^2 \\ T_3(s) = -s^{12} + 22\sqrt{2}s^9 + 22\sqrt{2}s^3 + 1 \end{gathered} \tag{6.14}$$

We find f and g from Formulas (3.8) and (1.8) by repeating the above procedure

$$f = (C_1 s + C_2)F_3{}^{-1/_6}, \qquad g = 2^{1/_3}5^{-1}F_3{}^{-5/_6}\, M_3(s)$$

$$M_3(s) = -\sqrt{2}\,C_2 s^5 + 5C_1 s^3 - 5C_2 s^2 - \sqrt{2}\,C_1 \tag{6.15}$$

Variable ζ is determined with the aid of relationships (6.15)

$$\zeta = G\frac{-\sqrt{2}\,s^5 + 5Es^3 - 5s^2 - \sqrt{2}E}{(Es+1)^5}, \qquad E = \frac{C_1}{C_2} \tag{6.16}$$

The integral curves in the $t\tau$-plane are easily found

$$t = \frac{5^2\,(2\sqrt{2}s + 7s^4 - 2\sqrt{2}s^7)\,(Es+1)^2}{(-\sqrt{2}s^5 + 5Es^3 - 5s^2 - \sqrt{2}E)^2}$$
$$\tau = -\frac{5^3}{3}\,\frac{(-s^{12} + 22\sqrt{2}s^9 + 22\sqrt{2}s^3 + 1)\,(Es+1)^3}{(-\sqrt{2}s^5 + 5Es^3 - 5s^2 - \sqrt{2}E)^3} \tag{6.17}$$

We also write down solution of the system (4.4)

$$U = (5G)^2\,\frac{2\sqrt{2}s + 7s^4 - 2\sqrt{2}s^7}{(Es+1)^8}, \quad V = -\frac{1}{3}(5G)^3\,\frac{-s^{12} + 22\sqrt{2}s^9 + 22\sqrt{2}s^3 + 1}{(Es+1)^{12}} \tag{6.18}$$

These equations have to be considered together with Equation (6.16). In order to separate from the general solution (6.17) that particular solution which defines the flow symmetric about the x-axis, it is necessary to find the real roots of Equation $T_3(s) = 0$. There are two such roots

$$s_1 = \sqrt{3} + \sqrt{2}, \qquad s_2 = \sqrt{2} - \sqrt{3}$$

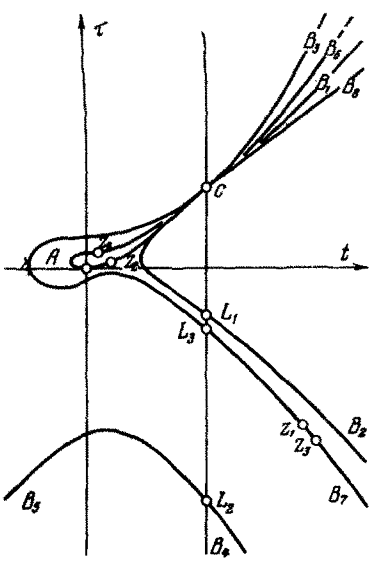

FIG. 7

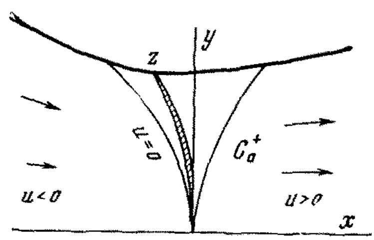

FIG. 8

We assume in Formulas (6.16) to (6.18) the constant E to be equal to $s = \sqrt{2} - \sqrt{3}$. The corresponding curve is shown on fig. 7. For $s = \sqrt{3} + \sqrt{2}$, Equations (6.17) yield the point A on the $t\tau$-plane. With the decrease of the parameter we move along the curve in the direction of $t < 0$. When $s = \sqrt{2}$, the curve intersects the line $t = 0$, which in the physical plane corresponds to transition through sonic velocity. The singular point C is reached when $s = 0.518$, and when the decreasing parameter passes through the values $s = 0.286$, the integral curve stretches into infinity along B_4, and then reappears along B_5. Line $t = 25$ is intersected when $s = 0.0673$. For $s = 0$ we have $t = 0$ and $\tau = -458.8$, and for $s \to s_2 = \sqrt{2} - \sqrt{3}$ the curve stretches to infinity along B_6 with an asymptotic behavior $\tau \sim 8t$. Decreasing the parameter still further results in the same curve being followed in the opposite direction, namely $B_6L_2B_5B_4CA$.

The solution derived in this section may be used for the analysis of certain special types of flow in Laval nozzles. We shall limit our analysis to flows in which subsonic and velocities exist simultaneously.

We may consider a flow pattern with weak discontinuities along the characteristics upstream and downstream of the nozzle center. One of such flows is represented in the $t\tau$-plane as follows: from point A of the curve shown on fig. 7 we move in the direction of $t > 0$ (the subsonic part of a Laval nozzle); we intersect line $t = 0$ (the sonic line) and reach point C (characteristic line C_0^-); then, instead of moving along the analytical continuation CB_4, we follow the curve CB_3 (along the characteristic C_0^-, fourth derivatives of the sonic stream perturbation velocity components will be discontinuous); we move into infinity along the branch B_3 (the y-axis in the real plane); then by an analytical continuation beyond the y-axis, we return along the curve B_3C to the point C (characteristic line C_0^+); from C we move to the point A along the curve reaching that point from the direction of $t > 0$ (the supersonic part of a Laval nozzle). A peculiarity of this flow is the symmetry of the stream about the y-axis in the area between the two characteristics.

We shall consider another possible application of the derived solution for the case of $n = 5$. Let us assume the presence of a small break in the wall of a Laval nozzle in its inlet part and in its supersonic zone, which widens the stream (point Z on fig. 8)

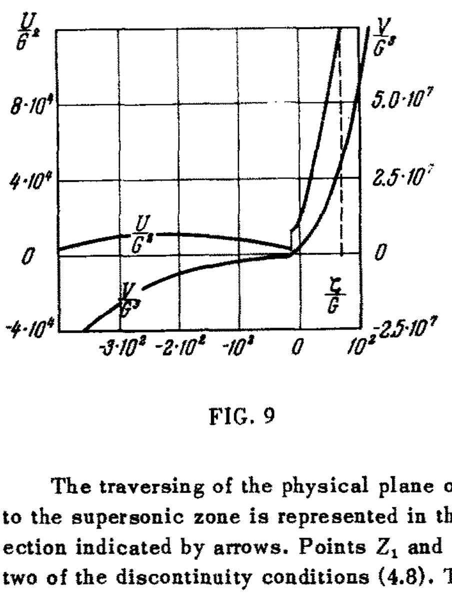

FIG. 9

A rarefied flow or the Prandtl—Meyer wave, spreads from this point. If the break is assumed to be small, the rarefied wave must be narrow. In this case the Prandtl — Meyer flow can be approximated by a rarefaction discontinuity. The suggestion to consider such discontinuities was first made by Frankl' in connection with the problem of a local supersonic zone [22]. It is interesting to note that two such discontinuities meeting at the center of a Laval nozzle, at its axis, need not produce a reflection, which shows that the flow along the C_0^+ characteristic is an analytic one. This means that it is possible to obtain in practice a sufficiently smooth supersonic stream in nonanalytic nozzles.

The traversing of the physical plane of the flow under consideration from the subsonic to the supersonic zone is represented in the $t\tau$-plane by curve $AZ_1Z_2B_2B_1CA$, in the direction indicated by arrows. Points Z_1 and Z_2 have been selected so, as to satisfy the first two of the discontinuity conditions (4.8). The third condition $\zeta_1=\zeta_2$ can be satisfied if the multiplication constant G is retained in Expressions (6.16) and (6.18), and made equal to 70.51G in Expressions (6.9) and (6.11), which define the flow downstream of the discontinuity. The position of discontinuity is determined by $\zeta/G = 13.34$. The behavior of the dimensionless velocity components U and V is shown on fig. 9.

7. We shall derive the solution for the case of $k = 1/30$ belonging to the family (2.6). We use the relevant solution obtained by Schwarz [6]

$$\xi = \frac{[H_4(s)]^3}{4^3 \cdot 3^3 [F_4(s)]^5}, \qquad H_4(s) = 1 + 228s^5 + 494s^{10} - 228s^{15} + s^{20}, \quad F_4(s) = s - 11s^6 - s^{11} \tag{7.1}$$

Polynomials $H_4(s)$ and $F_4(s)$ are such that the following equality is fulfilled

$$H_4^3 - 4^3 \cdot 3^3 F_4^5 = [T_4(s)]^2 \tag{7.2}$$
$$(T_4(s) = 1 - 522s^5 - 10005\, s^{10} - 10005 s^{20} + 522\, s^{25} + s^{30})$$

The solution of Equation (1.4) for $k = 1/30$ is found from (7.1) in conjunction with Equation

$$f = (C_1 s + C_2)\, F_4^{-1/12} \tag{7.3}$$

The corresponding solution is in the form of

$$g \sim F_4^{-3/4} M_4(s) \;((M_4(s) = C_2 - 11C_1 s - 66C_2 s^5 + 66C_1 s^6 - 11C_2 s^{10} + c_1 s^{11}) \tag{7.4}$$

The parametric form of solution of system (4.4) in the case of $n = 11$ considered here is

$$\begin{aligned} \zeta &= G\,(E - 11s - 66Es^5 + 66s^6 - 11Es^{10} + s^{11})\,(E + s)^{-11} \\ U &= (11G)^2\,(1 + 228s^5 + 494s^{10} - 228s^{15} + s^{20})\,(E + s)^{-20} \\ V &= {}^2/_3\,(11G)^3\,(1 - 522s^5 - 10005s^{10} - 10005s^{20} + 522s^{25} + s^{30})\,(E + s)^{-30} \end{aligned} \tag{7.5}$$

The corresponding solution in the $t\tau$-plane is found from Formulas (4.5). In order to separate from the general solution (7.5) the particular solution which defines the flow symmetric about the x-axis in the physical plane, we have to find the real roots of equation $T_4(s) = 0$. There are four of such roots

$$\begin{aligned} s_1 &= \frac{1}{2}\,[\sqrt{5} - 1 + \sqrt{10 - 2\sqrt{5}}], \qquad & s_2 &= \frac{1}{2}\,[\sqrt{5} - 1 - \sqrt{10 - 2\sqrt{5}}] \\ s_3 &= \frac{1}{2}\,[-\sqrt{5} - 1 + \sqrt{10 + 2\sqrt{5}}], \qquad & s_4 &= \frac{1}{2}\,[-\sqrt{5} - 1 - \sqrt{10 + 2\sqrt{5}}] \end{aligned} \tag{7.6}$$

The corresponding solution in the $t\tau$-plane will coincide with the solution indicated in (5.15), if in Formulas (7.5) we put $E = -s_1$. This solution gives part of the curve shown schematically on fig. 10.

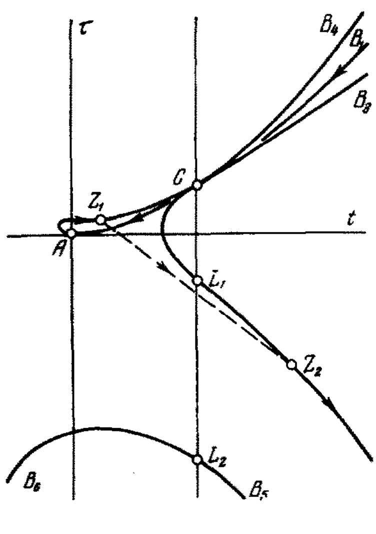

FIG. 10

When we decrease the parameter s from $s = s_1$, to $s = s_2$ the curve in the $t\tau$-plane runs from the point A in the direction of $t > 0$ along the path $ACB_1B_2L_1CB_3B_4L_2B_5$. A further decrease of this parameter results in this path being followed in the opposite direction.

We shall now obtain an analytical continuation of the plotted curve beyond the point A in the direction of $t < 0$.

In this case it is possible to do so without resorting to a new form of solutions. We assume $E = -s_3$ in Formulas (7.5). We decrease parameter s from $s = s_3$ to $s = s_4$ and plot in the $t\tau$-plane the corresponding curve which runs from the point A in the direction of $t < 0$ (fig. 10) along the path $ACB_6B_7L_3CB_8$. A further decrease of this parameter results in this path being traversed in the opposite direction.

It will be easily seen that the obtained solution viewed as a whole has no physical meaning, because of the presence of three limit lines L_1, L_2 and L_3. Lifschitz and Ryshov have suggested in [14] the introduction into the analysis of a shock wave $Z_1 Z_2$, and use of the following part of the plotted curve: from point A in the direction of $t < 0$ to point C, then along CB_6B_7, a jump from point Z_1 to point Z_2, with a return to point A from the direction of $t > 0$. The authors of [14] consider that this solution defines a certain asymptotic flow in the neighborhood a Laval nozzle center. The flow velocity along the axis of such a nozzle varies according to the law $u = \text{const}\ x^{20/11}$. Along the characteristic C_0^- upstream of the nozzle center the flow is analytic, except at the center itself, where $d^2u / dx^2 = \infty$. A shock wave is propagated through the stream from the nozzle center in the downstream direction, beyond which the stream is again accelerated. We may point out that the continuation of the given flow through the shock wave in such a way, as to maintain the stream symmetry about the x-axis beyond the shock wave, can be obtained in two ways, namely: by a jump from Z_1 to Z_2, or from Z_3 to Z_4 onto that branch of curve CA which approaches point A from the direction of $t < 0$.

In the second case the stream behind the shock wave continues to be decelerated, and becomes again subsonic.

The solution of system (4.4) in the case of $k = 1/30$ can also be expressed in another parametric form, different from (7.5), which may be more convenient for computations of certain problems. We shall consider solution (7.1) of the Schwarz problem. We carry out the linear transformation of function s.

$$s = -\frac{s' + a}{s'a - 1}, \qquad a = \frac{1}{4}\left[-2 - 2\sqrt{5} + \sqrt{10 - 2\sqrt{5}} + \sqrt{50 - 10\sqrt{5}}\right] \quad (7.7)$$

and obtain a new solution of the Schwarz equation in the form (the prime is omitted)

$$\xi = \frac{[H_5(s)]^3}{-{}^{27}/_5\,[F_5(s)]^5}$$

$$H_5(s) = -3\sqrt{5}s^{20} - 190s^{18} + 57\sqrt{5}s^{16} - 2280s^{14} + 1482\sqrt{5}s^{12} + {} + 4940s^{10} + 1482\sqrt{5}s^8 - 2280s^6 + 57\sqrt{5}s^4 - 190s^2 - 3\sqrt{5} \quad (7.8)$$

$$f_5(s) = \sqrt{5}s^{12} - 22s^{10} - 33\sqrt{5}s^8 + 44s^6 - 33\sqrt{5}s^4 - 22s^2 + \sqrt{5}$$

Instead of identity (7.2) we have the identity

$$H_5^3 + \frac{27}{5}F_5^5 = -\frac{4}{25}[T_5(s)]^2$$

$$T_5(s) = -225\sqrt{5}s^{29} - 2900s^{27} - 15921\sqrt{5}s^{25} + 104400s^{23} - 90045\sqrt{5}s^{21} - 200100s^{19} - {} - 570285\sqrt{5}s^{17} + 570285\sqrt{5}s^{13} + 200100s^{11} + 90045\sqrt{5}s^9 - 104400s^7 + {} + 15921\sqrt{5}s^5 + 2900s^3 + 225\sqrt{5}s \quad (7.9)$$

The solution of Equation (1.4) for $k = 1/30$ is then given by Equation (7.8) and

$$f = (C_1 s + C_2)F_5^{-1/12} \quad (7.10)$$

The solution of system (4.4) is then expressed by

$$\zeta = G(E + s)^{-11} M_5(s), \quad U = -(11G)^2 (E + s)^{-20} H_5, \quad V = \frac{4}{15}(11G)^3 (E + s)^{-30} T_5 \quad (7.11)$$

$$M_5(s) = -3\sqrt{5}Es^{11} - 11s^{10} + 55Es^9 - 33\sqrt{5}s^8 + 66\sqrt{5}Es^7 + 66s^6 - 66Es^5 - {} - 66\sqrt{5}s^4 + 33\sqrt{5}Es^3 - 55s^2 + 11Es + 3\sqrt{5}$$

It follows from the results obtained above that the solutions of system (4.4) for

$k = 1/6$, $1/12$, and $1/30$, when expressed in a parametric form, are single-valued functions of parameter s, in other words, s is the homogenizing variable. The generalised solution form is, in these cases, as follows

$$\zeta = GM_i(s)(E+s)^{-n}, \quad U = k_1 G^2 H_i(s)(E+s)^{-2n+2}, \quad V = k^2 G^3 T_i(s)(E+s)^{-3n+3}$$

where E, G, k_1, and k_2 are constants, and M_i, H_i, and T_i the corresponding polynomials.

8. We shall derive the solution for $k = 7/30$ in the family (2.7). Schwarz had shown in [6] that in this case the solution of Equation (3.6) can be expressed algebraically by the solution of that equation for $k = 1/30$. Klein [23] had found the form of this dependence. We denote the Schwarz function for the case of $k = 7/30$ by s_1, and the Schwarz function for $k = 1/30$, as before, by s. The following formula is then valid

$$s_1 = (s^7 + 7s^2)(7s^5 + 1)^{-1} \tag{8.1}$$

In order to find the solution of $f(\xi)$ we shall use Formula (3.8) where we substitute s_1 for s. We shall again use the parametric representation, but in terms of s, and not of s_1. We shall find $d\xi/ds_1 = d\xi/ds\, ds/ds_1$, where $d\xi/ds$ is defined by (7.1), and ds/ds_1 by (8.1)

$$\frac{d\xi}{ds} = -5 \cdot 4^{-3} \cdot 3^{-3} H_4^2 T_4 F_4^{-6}$$

Function $f(\xi)$ is defined by the formula

$$f = (C_1 s^7 + 7C_2 s^5 - 7C_1 s^2 + C_2) F^{-7/12}$$

together with Equation (7.1).

Having found the solution, we determine in the usual manner the solution of system (4.4) in the hodograph plane for $n = 17/7$

$$\zeta = \frac{7}{17} G \frac{s^{17} - 17Es^{15} + 119s^{12} + 187Es^{10} + 187s^7 - 119Es^5 + 17s^2 + E}{(s^7 + 7Es^5 - 7s^2 + E)^{17/7}}$$

$$U = G^2 H_4(s)(s^7 + 7Es^5 - 7s^2 + E)^{-20/7}, \quad V = \frac{2}{3} G^3 T_4(s)(s^7 + 7Es^5 - 7s^2 + E)^{-30/7}$$

The same solution for the case of $k = 7/30$ can be obtained in another form, if Equation (7.8) is used instead of (7.1).

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